

THE SINGULAR VALUES OF THE LOGARITHMIC POTENTIAL TRANSFORM ON BOUND STATES SPACES OF LANDAU HAMILTONIAN

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ABSTRACT. The singular values of the logarithmic potential transform on the generalized Bergmann space is calculated explicitly, too behavior in infinity.

1. INTRODUCTION

Let \mathbb{D} be the complex unit disk endowed with its Lebesgue measure μ and let $\partial\mathbb{D}$ be its boundary. Denote by $L^2(\mathbb{D}, d\mu)$ the space of complex-valued measurable functions which are $d\mu$ square integrable on \mathbb{D} . The logarithmic potential transform $: L^2(\mathbb{D}) \rightarrow L^2(\mathbb{D})$ is defined by

$$(1.1) \quad \mathcal{L}[f](z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} \log \left(\frac{1}{|z - \xi|} \right) d\mu(\xi)$$

This operator is very important as the transformed Cauchy and it often appears in Analysis [6].

The dimensional analysis [3, 6] and scaling arguments [5] form an integral part in theoretical physics to solve some important problems without doing much calculation.

The logarithmic potential in physics forms an interesting one as it provides some unusual prediction about the system. Moreover, this potential can be used suitably to illustrate some of the important features of field theory such as dimensional regularization and renormalization. In most of our text books, this potential is not discussed at detail; although the calculations are quite simple to demonstrate some of its unique features. We have obtained the bound state energy of this logarithmic potential through uncertainty principle, phase space quantization and Hellmann-Feynman theorem.

In [2] the authors have been dealing with the restriction of \mathcal{L} to the space $L_a^2(\mathbb{D})$ of analytic μ -square integrable on \mathbb{D} . They precisely have considered the projection operator $P_0: L^2(\mathbb{D}) \rightarrow L_a^2(\mathbb{D})$ and they have proved that the singular values λ_k of $\mathcal{L}P_0$, which turn out to be eigenvalues of the operator $\sqrt{(\mathcal{L}P_0)^*(\mathcal{L}P_0)}$ behave like k^{-1} as k goes to ∞ . They also concluded that $\mathcal{L}P_0$ belongs to the Schatten class $S_{1,\infty}$.

Now, consider the following *weighted* logarithmic potential transform

$$(1.1) \quad \mathcal{L}_\sigma[f](z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} \log \left(\frac{1}{|z - \xi|} \right) (1 - \xi\bar{\xi})^{\sigma-2} d\mu(\xi)$$

defined on the space $L^{2,\sigma}(\mathbb{D})$ of complex-valued measurable functions which are $(1-\xi\bar{\xi})^{\sigma-2}d\mu(\xi)$ -square integrable on \mathbb{D} where $\sigma > 1$ is a fixed parameter. We observe that the subspace $L_a^{2,\sigma}(\mathbb{D})$ of analytic functions on \mathbb{D} and belonging to $L^{2,\sigma}(\mathbb{D})$ coincides with the eigenspace

$$(1.3) \quad \mathcal{A}_0^\sigma(\mathbb{D}) := \left\{ \psi \in L^{2,\sigma}(\mathbb{D}), \Delta_\sigma \psi = 0 \right\}$$

of the second order differential operator

$$(1.4) \quad \Delta_\sigma := -4(1 - z\bar{z}) \left((1 - z\bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \sigma \bar{z} \frac{\partial}{\partial \bar{z}} \right)$$

known as the σ -weight Maass Laplacian and its discrete eigenvalues are given by

$$\epsilon_m := 4m(\sigma - 1 - m), \quad m = 0, 1, 2, \dots, \lfloor (\sigma - 1)/2 \rfloor$$

with their corresponding eigenspaces

$$(1.6) \quad \mathcal{A}_m^\sigma(\mathbb{D}) := \left\{ \psi \in L^{2,\sigma}(\mathbb{D}) \text{ and } \Delta_\sigma \psi = \epsilon_m^\sigma \psi \right\}$$

are here called *generalized Bergman spaces* since...

After noticing that, we here deal with analogous questions as in [2] in the context of the weighted Cauchy transform (1.2) and for its restriction to the space $\mathcal{A}_m^\sigma(\mathbb{D})$. That is, we are concerned with the operator $\mathcal{C}_\sigma P_m^\sigma$ where P_m^σ is the projection $L^{2,\sigma}(\mathbb{D}) \rightarrow \mathcal{A}_m^\sigma(\mathbb{D})$. The obtained results are as follows.

Firstly, we find that the singular values of $\mathcal{L}_\sigma P_m^\sigma$.

For $k \neq m$, can be expressed as

$$\lambda_k = \sqrt{J_1 + J_2 + J_3}$$

where

$$J_1 = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n+2k-2m+6-1)\Gamma(4\nu-2m-1)}{\Gamma(2n+2k-4m+4\nu+6)}$$

$$J_2 = \left(\frac{\alpha_k^{\nu,m}}{2\nu-m-1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu-2m-1)\Gamma(2n+2)}{\Gamma(2n+4\nu-2m+1)}$$

and

$$J_3 = \frac{(1+k-m)_m \alpha_k^{\nu,m}}{m!(k-m+1)(2\nu-m-1)} \left(\sum_{n=0}^{\infty} A_n \frac{\Gamma(k-m+2)\Gamma(4\nu-2m-1)}{\Gamma(4\nu-k-3m)} \right)$$

For $k = m$ can be expressed as

$$(1.2) \quad \lambda_k^2 = \frac{\alpha_k^{\nu,m} (2(2\nu-m)-1)}{8(\pi(2\nu-m+1))} \sum_{n=0}^{\infty} \frac{B_n}{n+2\nu-m}$$

where

$$B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2(\nu-m)+1)}{n!\Gamma(2(\nu-m))\Gamma(2\nu-m+2)}$$

$$\alpha_k^{\nu,m} = \frac{\Gamma(2)\Gamma(2(m-\nu)+1)}{\Gamma(m+1)\Gamma(2+m-2\nu)}$$

Secondly, we show that these singular values behave like

$$\lambda_k \sim C\sqrt{k^{m-4\nu+1}}, \text{ as } k \rightarrow \infty$$

where C is a constant.

The paper is organized as follows. In section 2, we review the definition of the weighted logarithmic potential transform as well as some of its needed properties. Section 3 deals with some basic facts on the spectral theory of Maass Laplacians on the Poincaré disk. In Section 4, a precise description of the generalized Bergmann spaces is reviewed. Section 5 is devoted to the computation of the singular values of the weighted logarithmic potential transform

The asymptotic behavior of these singular values is established in Section 6.

2. THE WEIGHTED LOGARITHMIC POTENTIAL TRANSFORM \mathcal{L}_ν

2.1. The case $\nu = 1$. Let \mathbb{D} the complex unit disk endowed with its Lebesgue measure μ and let $\partial\mathbb{D}$ its boundary denote by $L^2(\mathbb{D})$ the space of complex-valued measurable functions on \mathbb{D} with finite norm

$$(1.1) \quad \|f\| = \int_{\mathbb{D}} |f(\xi)|^2 d\mu(\xi)$$

The Logarithmic Potential operator $\mathcal{L} : L^2(\mathbb{D}) \rightarrow L^2(\mathbb{D})$ is defined by

$$(1.2) \quad \mathcal{L}[f](z) = \int_{\mathbb{D}} f(\xi) \log \left(\frac{1}{|\xi - z|} \right) d\mu(\xi)$$

2.2. The case of $\nu \geq 1$. We fix a real parameter ν such that $2\nu > 1$ and we consider the following weighted Logarithmic Potential transform

$$(1.4) \quad \mathcal{L}_\nu[f](z) = \int_{\mathbb{D}} f(\xi) \log \left(\frac{1}{|\xi - z|} \right) (1 - \xi\bar{\xi})^{2\nu-2} d\mu(\xi)$$

defined on the space $L^{2,\nu}(\mathbb{D})$ complex-valued measurable functions which are $(1 - \xi\bar{\xi})^{2\nu-2} d\mu(\xi)$ -square integrable on \mathbb{D} . As a convolution of $L^{2,\nu}$ -functions with the compactly supported measure $\frac{(1-\xi)^{2\nu-2}}{\xi} \llcorner_{\mathbb{D}} d\mu(\xi)$ $\mathcal{L}_\nu : L^{2,\nu}(\mathbb{D}) \rightarrow L^{2,\nu}(\mathbb{D})$ is obviously bounded. Moreover, it is not hard to show that \mathcal{L}_ν is in fact compact [1]. This raises a question concerning the spectral picture of \mathcal{L}_ν .

3. THE LANDAU HAMILTONIAN H_ν ON THE POINCARÉ DISK \mathbb{D}

Let $\mathbb{D} = \{z \in \mathbb{C}, z\bar{z} < 1\}$ be the complex unit disk with the Poincaré metric $ds^2 = 4(1 - z\bar{z})^{-2} dzd\bar{z}$. \mathbb{D} is a complete Riemannian manifold with all sectional curvature equal -1 . It has an ideal boundary $\partial\mathbb{D}$ identified with the circle $\{\omega \in \mathbb{C}, \omega\bar{\omega} = 1\}$. One refers to points $\omega \in \partial\mathbb{D}$ as points at infinity. The geodesic distance between two points z and w is given by

$$(2.1) \quad \cosh d(z, w) = 1 + \frac{2(z - w)(\bar{z} - \bar{w})}{(1 - z\bar{z})(1 - w\bar{w})}$$

By [10] the Schrödinger operator on \mathbb{D} with constant magnetic field of strength proportional to $\nu > 0$ can be written as :

$$(2.2) \quad \mathcal{L}_\nu := -(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \nu z (1 - |z|^2) \frac{\partial}{\partial z} + \nu \bar{z} (1 - |z|^2) \frac{\partial}{\partial \bar{z}} + \nu^2 |z|^2.$$

which is also called Maass Laplacian on the disk. A slight modification of \mathcal{L}_ν is given by the operator

$$(2.3) \quad H_\nu := 4\mathcal{L}_\nu - 4\nu^2$$

acting in the Hilbert space

$$(2.4) \quad L^{2,0}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \rightarrow \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 (1 - |z|^2)^{-2} d\mu(z) < +\infty \right\},$$

For our purpose, we shall consider the unitary equivalent realization \widetilde{H}_ν of the operator H_ν in the Hilbert space

$$(2.6) \quad L^{2,\nu}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \rightarrow \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 (1 - |z|^2)^{2\nu-2} d\mu(z) < +\infty \right\},$$

which is defined by

$$(2.7) \quad \widetilde{H}_\nu := \mathfrak{Q}_\nu^{-1} H_\nu \mathfrak{Q}_\nu,$$

where $\mathfrak{Q}_\nu : L^{2,\nu}(\mathbb{D}) \rightarrow L^{2,0}(\mathbb{D})$ is the unitary transformation defined by the map $\varphi \mapsto \mathfrak{Q}_\nu[\varphi] := (1 - |z|^2)^{-\nu} \varphi$. Different aspects of the spectral analysis of the operator \widetilde{H}_ν have been studied by many authors. For instance, note that \widetilde{H}_ν is an elliptic densely defined operator on the Hilbert space $L^{2,\nu}(\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by \widetilde{H}_ν . The spectrum of \widetilde{H}_ν in $L^{2,\nu}(\mathbb{D})$ consists of two parts: (i) a continuous part $[1, +\infty[$, which correspond to *scattering states*, (ii) a finite number of eigenvalues (*hyperbolic Landau levels*) of the form

$$(2.6) \quad \epsilon_m^\nu := 4(\nu - m)(1 - \nu + m), m = 0, 1, 2, \dots, \left[\nu - \frac{1}{2} \right]$$

with infinite degeneracy, provided that $2\nu > 1$. To the eigenvalues in (2.6) correspond eigenfunctions which are called *bound states* since the particle in such a state cannot leave the system without additionnal energy. A concrete description of these bound states spaces will be the goal of the next section.

4. THE BOUND STATES SPACES $\mathcal{A}_{\nu,m}^2(\mathbb{D})$

Here, we consider the eigenspace

$$(3.1) \quad \mathcal{A}_{\nu,m}^2(\mathbb{D}) := \left\{ \Phi : \mathbb{D} \rightarrow \mathbb{C}, \Phi \in L^{2,\nu}(\mathbb{D}) \text{ and } \widetilde{H}_\nu \Phi = \epsilon_m^\nu \Phi \right\}$$

See [9, 14], for the following proposition.

Proposition 1. *Let $2\nu > 1$ and $m = 0, 1, 2, \dots, \left[\nu - \frac{1}{2}\right]$. Then, we have*

(i) *an orthogonal basis of $\mathcal{A}_{\nu,m}^2(\mathbb{D})$ is given by the functions*

$$\begin{aligned} \phi_k^{\nu,m}(z) &:= |z|^{|m-k|} (1 - |z|^2)^{-m} e^{-i(m-k) \arg z} \\ &\times {}_2F_1 \left(-m + \frac{m-k+|m-k|}{2}, 2\nu - m + \frac{|m-k| - m + k}{2}, 1 + |m-k|; |z|^2 \right) \end{aligned}$$

$k = 0, 1, 2, \dots$, in terms of a terminating ${}_2F_1$ Gauss hypergeometric function.

(ii) *the norm square of $\phi_k^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by*

$$\|\phi_k^{\nu,m}\|^2 = \frac{\pi (\Gamma(1 + |m-k|))^2 \Gamma\left(m - \frac{|m-k|+m-k}{2} + 1\right) \Gamma\left(2\nu - m - \frac{|m-k|+m-k}{2}\right)}{(2(\nu - m) - 1) \Gamma\left(m + \frac{|m-k|-m+k}{2} + 1\right) \Gamma\left(2\nu - m + \frac{|m-k|-m+k}{2}\right)}.$$

Corollary 1. *The functions $\{\Phi_k^{\nu,m}\}$, $k = 0, 1, 2, \dots$ given by*

$$\begin{aligned} \Phi_k^{\nu,m}(z) &:= (-1)^k \left(\frac{2(\nu - m) - 1}{\pi} \right)^{\frac{1}{2}} \left(\frac{k! \Gamma(2(\nu - m) + m)}{m! \Gamma(2(\nu - m) + k)} \right)^{\frac{1}{2}} \\ &\times (1 - |z|^2)^{-m} \bar{z}^{m-k} P_k^{(m-k, 2(\nu-m)-1)}(1 - 2z\bar{z}) \end{aligned}$$

in terms of Jacobi polynomials constitute an orthonormal basis of $\mathcal{A}_m^{2,\nu}(\mathbb{D})$

Proof. Write the connection between the ${}_2F_1$ -sum and the Jacobi polynomial

$$P_k^{\alpha,\beta}(u) = \frac{(1+\alpha)_k}{k!} \cdot {}_2F_1(-k, 1+\alpha+\beta+k, 1+\alpha; \frac{1-u}{2})$$

then the functions

$$\phi_k^{\nu,m}(z) = \frac{(-1)^{\min(m,k)}}{(1 - |z|^2)^m} |z|^{|m-k|} e^{-i(m-k) \arg z} P_{\min(m,k)}^{(|m-k|, 2(\nu-m)-1)}(1 - 2z\bar{z})$$

constitute an orthonormal basis of $\mathcal{A}_{\nu,m}^2$. The norm square of $\phi_k^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by

$$\|\phi_k^{\nu,m}\|^2 = \frac{\pi}{(2(\nu - m) - 1)} \frac{(m \vee k)! \Gamma(2(\nu - m) + m \wedge k)}{(m \wedge k)! \Gamma(2(\nu - m) + m \vee k)}.$$

Here, $m \wedge k := \min(m, k)$ and $m \vee k := \max(m, k)$. Thus, the set of functions

$$\Phi_k^{\nu,m} := \frac{\phi_k^{\nu,m}}{\|\phi_k^{\nu,m}\|}, \quad k = 0, 1, 2, \dots$$

is an orthonormal basis of $\mathcal{A}_{\nu,m}^2(\mathbb{D})$ and can be rewritten as

$$(\star) \quad \Phi_k^{\nu,m}(z) = (-1)^k \left(\frac{2(\nu-m)-1}{\pi} \right)^{\frac{1}{2}} \left(\frac{k! \Gamma(2(\nu-m)+m)}{m! \Gamma(2(\nu-m)+k)} \right)^{\frac{1}{2}} \\ \times (1-|z|^2)^{-m} \bar{z}^{m-k} P_k^{(m-k, 2(\nu-m)-1)}(1-2z\bar{z})$$

by making appeal to the identity [12, p. 63]

$$\frac{\Gamma(m+1)}{\Gamma(m-s+1)} P_m^{(-s, \alpha)}(u) = \frac{\Gamma(m+\alpha+1)}{\Gamma(m-s+\alpha+1)} \left(\frac{u-1}{2} \right)^s P_{m-s}^{(s, \alpha)}(u), 1 \leq s \leq m$$

for $s = m - k$, $t = 1 - 2|z|^2$ and $\alpha = 2(\nu - m) - 1 \dots \square$

Corollary 2. *The L^2 -eigenspace $\mathcal{A}_{\nu,0}^2(\mathbb{D})$, corresponding to $m = 0$ in (3.1) and associated to the bottom energy $\epsilon'_0 = 0$ in (2.6), reduces further to the weighted Bergman space consisting of holomorphic functions $\phi : \mathbb{D} \rightarrow \mathbb{C}$ such that*

$$\int_{\mathbb{D}} |\phi(z)|^2 (1-|z|^2)^{2\nu-2} d\mu(z) < +\infty.$$

5. COMPUTATION OF THE SINGULAR VALUES λ_k

Elements of this basis are given in terms of Jacobi polynomials as

$$(5.1) \quad \phi_k^{\nu,m}(z) = \frac{(-1)^{\min(m,k)}}{(1-|z|^2)^m} |z|^{|m-k|} e^{-i(m-k)\arg z} P_{\min(m,k)}^{(|m-k|, 2(\nu-m)-1)}(1-2z\bar{z})$$

The norm square of $\phi_k^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by

$$(5.2) \quad \rho_k^{\nu,m} = \frac{\pi}{(2(\nu-m)-1)} \frac{(m \vee k)! \Gamma(2(\nu-m) + m \wedge k)}{(m \wedge k)! \Gamma(2(\nu-m) + m \vee k)}.$$

Here, $m \wedge k := \min(m, k)$ and $m \vee k := \max(m, k)$. Let us introduce the notation The set of functions

$$(5.3) \quad \gamma_k^{\nu,m} := \frac{(-1)^{m \wedge k}}{\sqrt{\rho_k^{\nu,m}}}, k = 0, 1, 2, \dots$$

So that we consider the elements

$$(4) \quad \Phi_k^{\nu,m}(z) := \gamma_k^{\nu,m} \frac{1}{(1-z\bar{z})^m} |z|^{|m-k|} e^{-i(m-k)\arg z} P_{\min(m,k)}^{(|m-k|, 2(\nu-m)-1)}(1-2z\bar{z})$$

5.1. The action \mathcal{L}_ν .

Lemma 1. We set $z = \rho e^{it}$, and $I = -\int_0^{2\pi} e^{i(k-m)\theta} \log(|z - re^{i\theta}|) \frac{d\theta}{2\pi}$, we have

$$(5.1) \quad \begin{cases} I = -\log(\rho \wedge r) & k = m \\ I = \frac{e^{i(k-m)t}}{2|m-k|} \left(\left(\frac{r}{\rho} \right)^{m-k} \wedge \left(\frac{r}{\rho} \right)^{m-k} \right) & k \neq m \end{cases}$$

Proof. By [2], it remain to prove that this lemma for $k < m$.

We have

$$\int_0^{2\pi} e^{i(k-m)\theta} \log(|\rho e^{it} - re^{i\theta}|) d\theta = -\int_0^{2\pi} e^{i(m-k)(-\theta)} \log(|re^{i(-t)} - \rho e^{i(-\theta)}|) d(-\theta)$$

The function $\theta \rightarrow e^{i(m-k)(-\theta)} \log(|re^{i(-t)} - \rho e^{i(-\theta)}|)$ is a periodic mapping with the period equal 2π , then

$$\begin{aligned} & \int_0^{2\pi} e^{i(k-m)\theta} \log(|\rho e^{it} - re^{i\theta}|) d\theta \\ &= -\int_0^{2\pi} e^{i(m-k)(-\theta)} \log(|re^{i(-t)} - \rho e^{i(-\theta)}|) d(-\theta) \\ &= \frac{e^{i(k-m)t}}{2(m-k)} \times \left(\left(\frac{r}{\rho} \right)^{m-k} \wedge \left(\frac{r}{\rho} \right)^{m-k} \right) \end{aligned}$$

□

Lemma 2. For all $\lambda \in \partial D$. \mathcal{L}_ν commutes with the rotations R_λ , where

$$(R_\lambda f)(z) = f(\lambda z)$$

Proof. We observe that

$$(R_\lambda \phi_k^{\nu,m})(z) = \lambda^{k-m} \phi_k^{\nu,m}(z), \quad \forall k \neq m$$

□

Corollary 3. $\{\mathcal{L}_\nu(\phi_k^{\nu,m})\}_{k=0}^\infty$ are orthonormal in $L^{2,\nu}(\mathbb{D})$

Proof. As R_λ is an isometry of $L^{2,\nu}(\mathbb{D})$,

$$\begin{aligned} & (\mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_j^{\nu,m})) \\ &= (R_\lambda \mathcal{L}_\nu(\phi_k^{\nu,m}), R_\lambda \mathcal{L}_\nu(\phi_j^{\nu,m})) \\ &= (\mathcal{L}_\nu R_\lambda(\phi_k^{\nu,m}), \mathcal{L}_\nu R_\lambda(\phi_j^{\nu,m})) \\ &= \overline{\lambda^{j-k}} (\mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_j^{\nu,m})), \quad \text{if } m > k \end{aligned}$$

or

$$= \lambda^{k-j} (\mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_j^{\nu,m})), \quad \text{if } m < k$$

For all $\lambda \in \partial D$, since $\lambda \neq 0$, we have

$$(\mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_j^{\nu,m})) = 0 \text{ if } j \neq k$$

□

Proposition 2. *The action of the operator \mathcal{L} on a basis element $\phi_k^{\nu,m}$, is of the form:
If $k = m$, We put $z = \rho e^{i\theta}$ then*

$$\mathcal{L}_\nu(\phi_k^{\nu,m})(z) = \beta(\rho) {}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m+1 \\ 2(\nu-m), 2\nu-m+2 \end{matrix} \mid 1-\rho^2 \right)$$

$$\text{with } \beta(\rho) = \frac{\alpha_k^{\nu,m}}{2(2\nu-m+1)} \sqrt{\frac{2(\nu-m)-1}{\pi}} (1-\rho^2)^{2\nu-m-1}.$$

If $k \neq m$ then

$$\mathcal{L}_\nu(\phi_k^{\nu,m})(z) = \frac{\pi \gamma_k^{\nu,m} e^{i(k-m)t}}{2(k-m)} (I_3 + I_4)$$

where

$$I_3 = \frac{(1+k-m)_m}{m!(k-m+1)} \rho^{k-m+2} (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2+k-m \end{matrix} \mid \rho^2 \right)$$

and

$$I_4 = \frac{\alpha_k^{\nu,m}}{2\nu-m-1} (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m-1 \\ 2(\nu-m), \end{matrix} \mid \rho^2 \right)$$

Proof. For $k = m$, we have

$$\begin{aligned} \mathcal{L}_\nu(\phi_k^{\nu,m})(z) &= \frac{(-1)^m}{\pi} \sqrt{\frac{2(\nu-m)-1}{\pi}} \\ &\int_{\mathbb{D}} (1-|\xi|^2)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2|\xi|^2) \log(|z-\xi|) d\mu(\xi) \\ &= (-1)^m \sqrt{\frac{2(\nu-m)-1}{\pi}} \int_0^1 (1-r^2)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2r^2) \log(\rho \wedge r) dr^2 \\ &= \frac{(-1)^m}{2} \sqrt{\frac{2(\nu-m)-1}{\pi}} \int_0^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \log(\rho^2 \vee t) dt \\ &= \frac{(-1)^m}{2} \sqrt{\frac{2(\nu-m)-1}{\pi}} [I_1 + I_2] \end{aligned}$$

Where

$$I_1 = \int_0^{\rho^2} (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \log(\rho^2 \vee t) dt$$

and

$$I_2 = \int_{\rho^2}^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \log(t) dt$$

Calculus of I_1 .

$$I_1 = \log(\rho^2) \int_{\rho^2}^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) dt$$

We use the formula

$$P_k^{(\alpha,\beta)}(u) = \frac{(1+\alpha)_k}{k!} {}_2F_1 \left(\begin{matrix} -k, 1+\alpha+\beta+k \\ 1+\alpha \end{matrix} \middle| \frac{1-u}{2} \right)$$

We have

$$I_1 = \log(\rho^2) \int_0^{\rho^2} (1-t)^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m, 2\nu-m \\ 1 \end{matrix} \middle| t \right) dt$$

By [11], we have

$$\int x^{c-1} (1-x)^{b-c-1} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) dx = \frac{1}{c} x^c (1-x)^{b-c} {}_2F_1 \left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \middle| x \right)$$

implies that

$$I_1 = \log(\rho^2) \rho^2 (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| \rho^2 \right)$$

Calculus of I_2 .

$$I_2 = \int_{\rho^2}^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \log(t) dt$$

Use the previous formula in [11] and the integration by part gives

$$\begin{aligned} I_2 &= \left[t(1-t)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) \log(t) \right]_{\rho^2}^1 \\ &\quad - \int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) dt \\ &= -\rho^2 \log(\rho^2) (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| \rho^2 \right) \\ &\quad - \int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) dt \end{aligned}$$

Calculus of

$$\int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) dt$$

Use the following formula which has place in [13]

$$\begin{aligned} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| t \right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1 \left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} \middle| 1-t \right) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-t)^{c-a-b} {}_2F_1 \left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} \middle| 1-t \right) \end{aligned}$$

We put $a = 1 - m$, $b = 2\nu - m$, $c = 2$ and use the formula Boher-Mollerup, for $z \in \mathbb{R}_+^*$,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{-\frac{z}{n}}$$

which implies $\frac{1}{\Gamma(1-m)} = 0$, then

$${}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) = \frac{2\Gamma(2(m-\nu)+1)}{m!\Gamma(2+m-2\nu)} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| 1-t \right)$$

implies that

$$\begin{aligned} & \int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) dt \\ &= \frac{2\Gamma(2(m-\nu)+1)}{m!\Gamma(2+m-2\nu)} \int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| 1-t \right) dt \end{aligned}$$

By the change $1-t=s$, we get

$$\begin{aligned} & \int_{\rho^2}^1 (1-t)^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| t \right) dt \\ &= \frac{2\Gamma(2(m-\nu)+1)}{m!\Gamma(2+m-2\nu)} \int_0^{1-\rho^2} t^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| t \right) dt \end{aligned}$$

In [11] page 44,

$$\int x^{\alpha-1} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| -t \right) dx = \frac{x^\alpha}{\alpha} {}_3F_2 \left(\begin{matrix} a, b, \alpha \\ c, \alpha+1 \end{matrix} \middle| -t \right) + \frac{\Gamma(\alpha)\Gamma(a-\alpha)\Gamma(b-\alpha)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-\alpha)}$$

Since $a = 1 - m$, $b = 2\nu - m$, $c = 2(\nu - m)$, and $\alpha = 2\nu - m + 1$ we have

$$\frac{\Gamma(\alpha)\Gamma(a-\alpha)\Gamma(b-\alpha)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-\alpha)} = 0$$

and by the change $t = -s$

$$\begin{aligned} & \int_0^{1-\rho^2} t^{2\nu-m+1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| t \right) dt \\ &= (-1)^m \int_0^{\rho^2} t^{2\nu-m} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| -t \right) dt \\ &= (-1)^m \frac{(\rho^2-1)^{2\nu-m+1}}{2\nu-m+1} {}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m+1 \\ 2(\nu-m), 2\nu-m+2 \end{matrix} \middle| 1-\rho^2 \right) \end{aligned}$$

we set $\alpha_k^{\nu,m} = \frac{2\Gamma(2(m-\nu)+1)}{m!\Gamma(2+m-2\nu)}$. We get

$$I_2 = -\rho^2 \log(\rho^2) (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2 \end{matrix} \middle| \rho^2 \right)$$

$$+(-1)^m \alpha_k^{\nu,m} \frac{(1-\rho^2)^{2\nu-m-1}}{2\nu-m+1} {}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m+1 \\ 2(\nu-m), 2\nu-m+2 \end{matrix} \mid 1-\rho^2 \right)$$

Finally

$$\begin{aligned} \mathcal{L}_\nu(\phi_k^{\nu,m})(z) &= \frac{\alpha_k^{\nu,m}}{2(2\nu-m+1)} \sqrt{\frac{2(\nu-m)-1}{\pi}} (1-\rho^2)^{2\nu-m-1} \\ &\quad \times {}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m+1 \\ 2(\nu-m), 2\nu-m+2 \end{matrix} \mid 1-\rho^2 \right) \end{aligned}$$

Now if $k > m$, set $z = \rho e^{it}$.

$$\begin{aligned} \mathcal{L}_\nu(\phi_k^{\nu,m})(z) &= \gamma_k^{\nu,m} \int_{\mathbb{D}} (1-|\xi|^2)^{2\nu-m-2} \xi^{k-m} \log \left(\frac{1}{|z-\xi|} \right) P_m^{(k-m, 2(\nu-m)-1)} (1-2|\xi|^2) d\mu(\xi) \\ &= \gamma_k^{\nu,m} \int_0^1 (1-r^2)^{2\nu-m-2} r^{k-m+1} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \int_0^{2\pi} e^{i(k-m)\theta} \log \left(\frac{1}{|z-r^{i\theta}|} \right) d\theta dr \\ &= \frac{\pi \gamma_k^{\nu,m} e^{i(k-m)t}}{2(k-m)} \int_0^1 (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \left(\left(\frac{r}{\rho} \right)^{k-m} \wedge \left(\frac{\rho}{r} \right)^{k-m} \right) dr^2 \\ &= \frac{\pi \gamma_k^{\nu,m} e^{i(k-m)t}}{2(k-m)} \left(\int_0^\rho (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \left(\left(\frac{r}{\rho} \right)^{k-m} \wedge \left(\frac{\rho}{r} \right)^{k-m} \right) dr^2 \right. \\ &\quad \left. + \int_\rho^1 (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \left(\left(\frac{r}{\rho} \right)^{k-m} \wedge \left(\frac{\rho}{r} \right)^{k-m} \right) dr^2 \right) \end{aligned}$$

We set

$$I_3 = \int_0^\rho (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \left(\left(\frac{r}{\rho} \right)^{k-m} \wedge \left(\frac{\rho}{r} \right)^{k-m} \right) dr^2$$

and

$$I_4 = \int_\rho^1 (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2r^2) \left(\left(\frac{r}{\rho} \right)^{k-m} \wedge \left(\frac{\rho}{r} \right)^{k-m} \right) dr^2$$

Calculus of I_3 .

$$I_3 = \frac{\rho^{m-k}(1+k-m)_m}{m!} \int_0^{\rho^2} t^{k-m} (1-t)^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m, 2(\nu-m)+k \\ 1+k-m \end{matrix} \mid t \right) dt$$

By the formula

$$\int x^{c-1} (1-x)^{b-c-1} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \mid x \right) dx = \frac{1}{c} x^c (1-x)^{b-c} {}_2F_1 \left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \mid x \right)$$

we have

$$I_3 = \frac{(1+k-m)_m}{m!(k-m+1)} \rho^{k-m+2} (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2+k-m \end{matrix} \mid \rho^2 \right)$$

Calculus of I_4 .

$$\begin{aligned} I_4 &= \int_{\rho}^1 (1-r^2)^{2\nu-m-2} r^{k-m} P_m^{(k-m, 2(\nu-m)-1)} \left(1-2r^2\right) \left(\left(\frac{r}{\rho}\right)^{k-m} \wedge \left(\frac{\rho}{r}\right)^{k-m}\right) dr^2 \\ &= \frac{\rho^{k-m}(1+k-m)_m}{2m!} \int_{\rho^2}^1 (1-t)^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m, 2(\nu-m)+k \\ 1+k-m \end{matrix} \middle| t \right) dt \end{aligned}$$

As the previous

$$\begin{aligned} &\int_{\rho^2}^1 (1-t)^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m, 2(\nu-m)+k \\ 1+k-m \end{matrix} \middle| t \right) dt \\ &= \alpha_k^{\nu, m} \int_{\rho^2}^1 (1-t)^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| 1-t \right) dt \\ &= (-1)^m \alpha_k^{\nu, m} \int_{\rho^2-1}^0 t^{2\nu-m-2} {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m \\ 2(\nu-m) \end{matrix} \middle| t \right) dt \\ &= \frac{\alpha_k^{\nu, m}}{2\nu-m-1} (1-\rho^2)^{2\nu-m-1} {}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m-1 \\ 2(\nu-m), 2\nu-m \end{matrix} \middle| 1-\rho^2 \right) \end{aligned}$$

also

$${}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2\nu-m-1 \\ 2(\nu-m), 2\nu-m \end{matrix} \middle| 1-\rho^2 \right) = {}_2F_1 \left(\begin{matrix} -m+1, 2\nu-m-1 \\ 2(\nu-m) \end{matrix} \middle| 1-\rho^2 \right)$$

Now if $k < m$. We have

$$\phi_k^{\nu, m}(z) = (-1)^k \sqrt{\frac{2(\nu-m)-1}{\pi} \frac{k! \Gamma(2(\nu-m)+m)}{m! \Gamma(2(\nu-m)+k)}} (1-|z|^2)^{-m} \bar{z}^{m-k} P_k^{(m-k, 2(\nu-m)-1)} (1-2|z|^2)$$

By the formula

$$\frac{\Gamma(m+1)}{\Gamma(m-s+1)} P_m^{(-s, \alpha)}(u) = \frac{\Gamma(m+\alpha+1)}{\Gamma(m-s+\alpha+1)} \left(\frac{u-1}{2}\right)^s P_{m-s}^{(s, \alpha)}(u), 1 \leq s \leq m$$

and put $s = m - k$ and $\alpha = 2(\nu - m) - 1$, we have

$$P_k^{(m-k, 2(\nu-m)-1)} (1-2|z|^2) = \frac{m! \Gamma(k+\alpha+1)}{k! \Gamma(m+\alpha+1)} P_m^{(k-m, 2(\nu-m)-1)} (1-2|z|^2)$$

substituting in the expression of $\phi_k^{\nu, m}(z)$, we get

$$\phi_k^{\nu, m}(z) = (-1)^m \sqrt{\frac{2(\nu-m)-1}{\pi} \frac{m! \Gamma(2(\nu-m)+k)}{k! \Gamma(2(\nu-m)+m)}} (1-|z|^2)^{-m} z^{k-m} P_m^{(k-m, 2(\nu-m)-1)} (1-2|z|^2)$$

it's the same formula for $k > m$, which prove the same formula of $\mathcal{L}_{\nu}(\phi_k^{\nu, m})(z)$ if $k > m$. \square

Remark 1. By the previous formula in [13], we have

$${}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2(\nu-m) \end{matrix} \middle| \rho^2 \right) = \frac{k! \Gamma(2+k-m)}{\Gamma(1-2(\nu-m))} {}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2(\nu-m) \end{matrix} \middle| 1-\rho^2 \right)$$

5.2. The spectrum of \mathcal{L}_ν .

Proposition 3. *If $k \neq m$, then*

$$\lambda_k = \sqrt{J_1 + J_2 + J_3}$$

where

$$J_1 = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n+2k-2m+6-1)\Gamma(4\nu-2m-1)}{\Gamma(2n+2k-4m+4\nu+6)}$$

$$J_2 = \left(\frac{\alpha_k^{\nu,m}}{2\nu-m-1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu-2m-1)\Gamma(2n+2)}{\Gamma(2n+4\nu-2m+1)}$$

and

$$J_3 = \frac{(1+k-m)_m \alpha_k^{\nu,m}}{m!(k-m+1)(2\nu-m-1)} \left(\sum_{n=0}^{\infty} A_n \frac{\Gamma(k-m+2)\Gamma(4\nu-2m-1)}{\Gamma(4\nu-k-3m)} \right)$$

If $k = m$ then

$$(5.2) \quad \lambda_k^2 = \frac{\alpha_k^{\nu,m} (2(2\nu-m)-1)}{8(\pi(2\nu-m+1))} \sum_{n=0}^{\infty} \frac{B_n}{n+2\nu-m}$$

where

$$B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2(\nu-m)+1)}{n!\Gamma(2(\nu-m))\Gamma(2\nu-m+2)}$$

Proof. If $k \neq m$. We have

$$(\mathcal{L}_\nu(\phi_k^{\nu,m}))(z) = \frac{\pi \gamma_k^{\nu,m} (I_3 + I_4)}{2(k-m)} e^{i(k-m)t}$$

We set $\mathcal{H} = (L^2(\mathbb{D}), (1-|\xi|^2)^{2\nu-2} d\mu(\xi))$, $I_3 = I_3(\rho)$, and $I_4 = I_4(\rho)$ we have

$$\begin{aligned} \lambda_k^2 &= \langle \mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_k^{\nu,m}) \rangle_{\mathcal{H}} \\ &= \frac{\pi^2 \gamma_k^{\nu,m}}{(k-m)} \int_0^1 (I_3(\rho) + I_4(\rho))^2 \rho d\rho \end{aligned}$$

Calculus of $\int_0^1 (I_3(\rho))^2 \rho d\rho$.

$$I_3(\rho) = \frac{(1+k-m)_m}{m!(k-m+1)} \rho^{k-m+2} (1-\rho^2)^{2\nu-m-1} {}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2+k-m \end{matrix} \middle| \rho^2 \right)$$

Since

$${}_2F_1 \left(\begin{matrix} -m+1, 2(\nu-m)+k \\ 2+k-m \end{matrix} \middle| \rho^2 \right) = \sum_{n=0}^{\infty} \frac{(-m+1)_n (2(\nu-m)+k)_n}{(2+k-m)_n} \frac{\rho^{2n}}{n!}$$

then

$$(I_3(\rho))^2 = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \rho^{2n} (1-\rho^2)^{4\nu-2m-2}$$

where

$$A_n = \frac{1}{n!} \sum_{i=0}^n \frac{(-m+1)_i (-m+1)_{n-i} (2(\nu-m)+k)_i (2(\nu-m)+k)_{n-i}}{(2(\nu-m))_i (2(\nu-m))_{n-i}}$$

Thus

$$J_1 = \int_0^1 (I_3(\rho))^2 \rho d\rho = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \int_0^1 \rho^{2n+2k-2m+6-1} (1-\rho^2)^{4\nu-2m-1-1} d\rho$$

Use the fact that

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

implies

$$(5.3) \quad \int_0^1 (I_3(\rho))^2 \rho d\rho = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n+2k-2m+6-1)\Gamma(4\nu-2m-1)}{\Gamma(2n+2k-4m+4\nu+6)}$$

Calculus of $\int_0^1 (I_4(\rho))^2 \rho d\rho$.

In the same

$$(5.4) \quad J_2 = \int_0^1 (I_4(\rho))^2 \rho d\rho = \left(\frac{\alpha_k^{\nu,m}}{2\nu-m-1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu-2m-1)\Gamma(2n+2)}{\Gamma(2n+4\nu-2m+1)}$$

Calculus of $2 \int_0^1 (I_3(\rho)) (I_4(\rho)) \rho d\rho$.

$$(5.5) \quad J_3 = 2 \int_0^1 (I_3(\rho)) (I_4(\rho)) \rho d\rho = \frac{(1+k-m)_m \alpha_k^{\nu,m}}{m!(k-m+1)(2\nu-m-1)} \sum_{n=0}^{\infty} A_n \frac{\Gamma(k-m+2)\Gamma(4\nu-2m-1)}{\Gamma(4\nu-k-3m)}$$

If $k = m$.

Since

$$\begin{aligned} & \left({}_3F_2 \left(\begin{matrix} -m+1, 2\nu-m, 2(\nu-m)+1 \\ 2(\nu-m), 2\nu-m+2 \end{matrix} \mid 1-\rho^2 \right) \right)^2 \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2(\nu-m)+1)}{n!\Gamma(2(\nu-m))\Gamma(2\nu-m+2)} (1-\rho^2)^n \\ \lambda_k^2 &= \frac{\alpha_k^{\nu,m} (2(2\nu-m)-1)}{8(\pi(2\nu-m+1))} \sum_{n=0}^{\infty} B_n \int_0^1 (1-\rho^2)^{n+2\nu-m-1} d\rho \end{aligned}$$

$$= \frac{\alpha_k^{\nu,m} (2(2\nu - m) - 1)}{8(\pi(2\nu - m + 1))} \sum_{n=0}^{\infty} \frac{B_n}{n + 2\nu - m}$$

where

$$B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2(\nu-m)+1)}{n!\Gamma(2(\nu-m))\Gamma(2\nu-m+2)}$$

□

6. ASYMPTOTIC BEHAVIOR OF SINGULAR VALUES λ_k AS $k \rightarrow \infty$

Proposition 4.

$$\lambda_k \sim C\sqrt{k^{m-4\nu+1}}, \text{ as } k \rightarrow \infty$$

where C is a constant

Proof. If $k > m$, then

$$\lambda_k = \sqrt{J_1 + J_2 + J_3}$$

where

$$J_1 = \left(\frac{(1+k-m)_m}{m!(k-m+1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n+2k-2m+6-1)\Gamma(4\nu-2m-1)}{\Gamma(2n+2k-4m+4\nu+6)}$$

$$J_2 = \left(\frac{\alpha_k^{\nu,m}}{2\nu-m-1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu-2m-1)\Gamma(2n+2)}{\Gamma(2n+4\nu-2m+1)}$$

and

$$J_3 = \frac{(1+k-m)_m \alpha_k^{\nu,m}}{m!(k-m+1)(2\nu-m-1)} \left(\sum_{n=0}^{\infty} A_n \frac{\Gamma(k-m+2)\Gamma(4\nu-2m-1)}{\Gamma(4\nu-k-3m)} \right)$$

The limit of λ_k as $k \rightarrow \infty$.

We use the formula

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} \sim k^{a-b}$$

we have

(6.1)

$$J_1 \sim \left(\frac{k^{-1-m}}{m!} \right)^2 \sum_{n=0}^{\infty} A_n \Gamma(4\nu-2m-1)(2k)^{2m-4\nu-1} \sim k^{-4\nu-1} 2^{2m-4\nu-1} \Gamma(4\nu-2m-1) \sum_{n=0}^{\infty} \frac{A_n}{m!}$$

(6.2)

$$J_2 = \mathcal{O}_{k \sim \infty}(1)$$

In the same

(6.3)

$$J_3 \sim k^{m-4\nu+1} \frac{\alpha_k^{\nu,m} \Gamma(4\nu-2m-1)}{m!(2\nu-m-1)} \sum_{n=0}^{\infty} A_n$$

Therefore

$$\lambda_k \sim C\sqrt{k^{m-4\nu+1}}$$

where C is a constant

□

REFERENCES

- [1] : Anderson, J.M.; Khavinson, D. and Lomonosov, V. Spectral properties of some operators arising in potential theory, 1990, preprint, 29pp.
- [2] : J.Arazy , D.Khavinson spectral estimates of Cauchy's transform in $L^2(\Omega)$. Integr Equat Oper Th. Vol. 15 (1992).
- [3] : G. I. Barenblatt Dimensional Analysis, (Gordon Breach Science Publishers, New York, 1987).
- [4] : D. Jana Phys. Edu.,25, 35 (2008).
- [5] : D. Jana Phys. Edu.,19, 167 (2002).
- [6] : H. L. Langhaar. Dimensional Analysis and theory of models, (Wiley, New York,1951).
- [7] : Multin R. Dostanic. The properties of Cauchy transform on a bounded domain. J.operator theory 36(1996), 233-247.
- [8] : Multin R. Dostanic. Norm estimate of the Cauchy transform on $L^p(\Omega)$.
- [9] :Mourad E.H.Ismail, Classical and Quantum Orthogonal Polynomials in one variable, Encyclopedia of Mathematics and its applications, Cambridge university press(2005).
- [10] : Ferapontov, E.V., Veselov, A.P. Integrable Schrodinger operators with magnetic fields: factorization method on curved surfaces. J. Math. Phys. 42, 590-607 (2001).
- [11] : A.P.Prudnikov , Yu.A.Brychkov ,O.I.Marichev . Integrals and Series Volume 3 More spacial functions Gordon and Breach , New York , 1990.
- [12] : A. Szegö, Orthogonal polynomials. American Mathematical Society; Providence, R.I. (1975).
- [13] :H. Srivastava and L. Manocha, A Treatise on Generating Functions, Ellis Horwood Ltd, London 1984.
- [14] :W.Magnus, F.Oberhettinger and R.P.Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag Berlin Heidelberg New york, 1966.

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